9. *Nested Squares.* A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 49). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

54. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

(1) $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}.$$

Expression (1) is called the ***Cauchy integral formula***. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f *on* C.

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 68). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 53) that

$$\int_C \frac{f(z)\,dz}{z - z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z - z_0}.$$

This enables us to write

(2) $$\int_C \frac{f(z)\,dz}{z - z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

But [see Exercise 13, Sec. 46]

$$\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i,$$

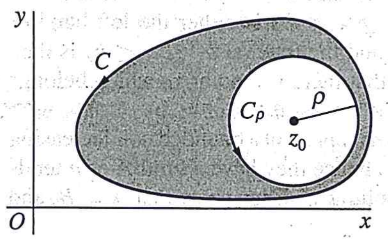

FIGURE 68

and so equation (2) becomes

(3) $$\int_C \frac{f(z)\,dz}{z - z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

(4) $$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z - z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (4) holds when z is such a point; and the theorem in Sec. 47, giving upper bounds for the moduli of contour integrals, tells us that

$$\left|\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon.$$

In view of equation (3), then,

$$\left|\int_C \frac{f(z)\,dz}{z - z_0} - 2\pi i f(z_0)\right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it follows that

$$\int_C \frac{f(z)\,dz}{z - z_0} - 2\pi i f(z_0) = 0.$$

Hence equation (1) is valid, and the theorem is proved.

When the Cauchy integral formula is written as

(5) $$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0),$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 1$ about the origin. Since the function

$$f(z) = \frac{\cos z}{z^2 + 9}$$

is analytic inside and on C and since the origin $z_0 = 0$ is interior to C, equation (5) tells us that

$$\int_C \frac{\cos z}{z(z^2 + 9)}\,dz = \int_C \frac{(\cos z)/(z^2 + 9)}{z - 0}\,dz = 2\pi i f(0) = \frac{2\pi i}{9}.$$

55. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives $f^{(n)}(z_0)$ of f at z_0.

Theorem. *Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots), \tag{1}$$

With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

this theorem includes the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}. \tag{2}$$

Verification of expression (1) will be taken up in Sec. 56.

When written in the form

$$\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \quad (n = 0, 1, 2, \ldots), \tag{3}$$

expression (1) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (3) shows that

$$\int_C \frac{dz}{z-z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = 0 \quad (n = 1, 2, \ldots).$$

(Compare with Exercise 13, Sec. 46.)

Expression (1) can also be useful in slightly different notation. Namely, if s denotes points on C and if z is a point interior to C, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots), \tag{4}$$

where $f^{(0)}(z) = f(z)$ and, of course, $0! = 1$. Our next example illustrates the use of expression (4) in the form

$$\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z) \quad (n = 0, 1, 2, \ldots), \tag{5}$$

which includes the special case

$$\int_C \frac{f(s)\,ds}{s-z} = 2\pi i\, f(z). \tag{6}$$

EXAMPLE 3. If n is a nonnegative integer and $f(z) = (z^2-1)^n$, expression (4), becomes

$$\frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots), \tag{7}$$

where C is any simple closed contour surrounding z. In view of equation (7), one can write the Legendre polynomial*

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n \quad (n = 0, 1, 2, \ldots) \tag{8}$$

as

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots). \tag{9}$$

Because

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1},$$

expression (9) reveals that

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s+1)^n ds}{s-1} \quad (n = 0, 1, 2, \ldots);$$

and by writing $f(s) = (s+1)^n$ and $z = 1$ in equation (6), we arrive at the values

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\, 2\pi i\,(1+1)^n = 1 \quad (n = 0, 1, 2, \ldots).$$

The values $P_n(-1) = (-1)^n (n = 0, 1, 2, \ldots)$ can be found (Exercise 8, Sec. 57) in a similar way.

*See Exercise 10, Sec. 20, and the footnote with it.

Finally, we note how expression (4) is suggested. If s denotes points on C and z is a point interior to C, the Cauchy integral formula is

(10) $$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}.$$

By differentiating formally under this integral sign, without rigorous justification, we find that

$$f'(z) = \frac{1}{2\pi i}\int_C f(s)\frac{\partial}{\partial z}(s-z)^{-1}\,ds,$$

or

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}.$$

Likewise,

$$f''(z) = \frac{(2)(1)}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{2+1}}$$

and

$$f'''(z) = \frac{(3)(2)(1)}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{3+1}}.$$

These three special cases suggest that expression (4), which is to be verified in Sec. 56, *may* be valid. A reader who wishes to accept expression (4) without verification can easily pass to Sec. 57.

56. VERIFICATION OF THE EXTENSION

We turn now to the verification of the extended Cauchy integral formula that was introduced in Sec. 55. Specifically, we consider a function f that is analytic inside and on a simple closed contour C, taken in the positive sense, and we let z be any point interior to C. We begin with statement (10), Sec. 55, of the Cauchy integral formula:

(1) $$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}.$$

In order to verify that $f'(z)$ exists and that the expression

(2) $$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

in Sec. 55 is valid, we let d denote the smallest distance from z to points s on C and assume that $0 < |\Delta z| < d$ (see Fig. 69). It then follows from expression (1) that

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds.$$

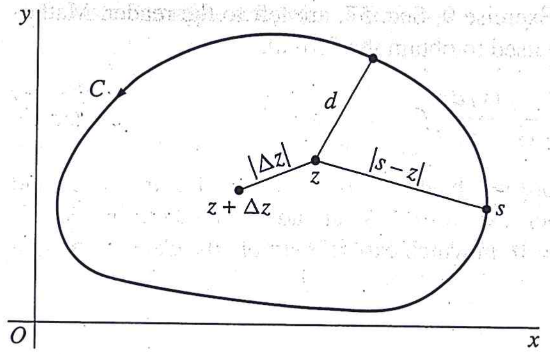

FIGURE 69

Evidently, then,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)\,(s-z)}.$$

But

$$\frac{1}{(s-z-\Delta z)\,(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)\,(s-z)^2},$$

and this means that

(3) $$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}.$$

Next, we let M denote the maximum value of $|f(s)|$ on C and observe that since $|s-z| \ge d$ and $|\Delta z| < d$,

$$|s-z-\Delta z| = |(s-z)-\Delta z| \ge ||s-z|-|\Delta z|| \ge d-|\Delta z| > 0.$$

Thus

$$\left|\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{|\Delta z|M}{(d-|\Delta z|)d^2}L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (3) also tends to zero. Consequently,

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = 0;$$

and the desired expression for $f'(z)$ is established.

The same technique can be used to verify the expression

(4) $$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}.$$

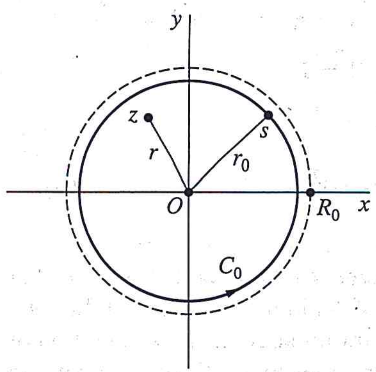

FIGURE 78

Now the factor $1/(s-z)$ in the integrand here can be put in the form

(2) $$\frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{1-(z/s)};$$

and we know from the example in Sec. 56 that

(3) $$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$$

when z is any complex number other than unity. Replacing z by z/s in expression (3), then, we can rewrite equation (2) as

(4) $$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(s-z)s^N}.$$

Multiplying through this equation by $f(s)$ and then integrating each side with respect to s around C_0, we find that

$$\int_{C_0} \frac{f(s)\,ds}{s-z} = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} z^n + z^N \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}.$$

In view of expression (1) and the fact that (Sec. 55)

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

this reduces, after we multiply through by $1/(2\pi i)$, to

(5) $$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z),$$

where

(6) $$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}.$$

Representation (4) in Sec. 62 now follows once it is shown that

(7) $$\lim_{N\to\infty} \rho_N(z) = 0.$$

To accomplish this, we recall that $|z| = r$ and that C_0 has radius r_0, where $r_0 > r$. Then, if s is a point on C_0, we can see that

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Consequently, if M denotes the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} 2\pi r_0 = \frac{Mr_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N.$$

Inasmuch as $(r/r_0) < 1$, limit (7) clearly holds.

The case $z_0 \neq 0$

In order to verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that f is analytic when $|z - z_0| < R_0$ and note that the composite function $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < R_0$. This last inequality is, of course, just $|z| < R_0$; and, if we write $g(z) = f(z + z_0)$, the analyticity of g in the disk $|z| < R_0$ ensures the existence of a Maclaurin series representation:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \qquad (|z| < R_0).$$

That is,

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \qquad (|z| < R_0).$$

After replacing z by $z - z_0$ in this equation and its condition of validity, we have the desired Taylor series expansion (1) in Sec. 62.

64. EXAMPLES

In Sec. 72, we shall see that any Taylor series representing a function $f(z)$ about a given point z_0 is unique. More precisely, we will show that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all points z interior to some circle centered at z_0, then the power series here must be *the* Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways than by appealing directly to the formula $a_n = f^{(n)}(z_0)/n!$ in Taylor's theorem.

This section is devoted to finding the following six Maclaurin series expansions, where $z_0 = 0$, and to illustrate how they can be used to find related expansions:

(1) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1),$$

(2) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad (|z| < \infty),$$

(3) $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad (|z| < \infty),$$

(4) $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad (|z| < \infty),$$

(5) $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad (|z| < \infty),$$

(6) $$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad (|z| < \infty).$$

We list these results together in order to have them for ready reference later on. Since the expansions are familiar ones from calculus with z instead of x, the reader should, however, find them easy to remember.

In addition to collecting expansions (1) through (6) together, we now present their derivations as Examples 1 through 6, along with some other series that are immediate consequences. The reader should always keep in mind that

(*a*) the regions of convergence can be determined before the actual series are found;
(*b*) there may be several reasonable ways to find the desired series.

EXAMPLE 1. Representation (1) was, of course, obtained earlier in Sec. 61, where Taylor's theorem was not used. In order to see how Taylor's theorem can be used, we first note that the point $z = 1$ is the only singularity of the function

$$f(z) = \frac{1}{1-z}$$

in the finite plane. So the desired Maclaurin series converges to $f(z)$ when $|z| < 1$.

The derivatives of $f(z)$ are

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \qquad (n = 1, 2, \ldots).$$

Hence if we agree that $f^{(0)}(z) = f(z)$ and $0! = 1$, we find that $f^{(n)}(0) = n!$ when $n = 0, 1, 2, \ldots$; and upon writing

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n,$$

we arrive at the series representation (1).

If we substitute $-z$ for z in equation (1) and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z| < 1).$$

If, on the other hand, we replace the variable z in equation (1) by $1 - z$, we have the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1).$$

This condition of validity follows from the one associated with expansion (1) since $|1 - z| < 1$ is the same as $|z - 1| < 1$.

For another application of expansion (1), we now seek a Taylor series representation of the function

$$f(z) = \frac{1}{1-z}$$

about the point $z_0 = i$. Since the distance between z_0 and the singularity $z = 1$ is $|1 - i| = \sqrt{2}$, the condition of validity is $|z - i| < \sqrt{2}$. (See Fig. 79.) To find the series, which involves powers of $z - i$, we first write

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - \left(\dfrac{z-i}{1-i}\right)}.$$

Because

$$\left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{|1-i|} = \frac{|z-i|}{\sqrt{2}} < 1$$

when $|z - i| < \sqrt{2}$, expansion (1) now tells us that

$$\frac{1}{1 - \left(\dfrac{z-i}{1-i}\right)} = \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n \qquad (|z-i| < \sqrt{2});$$

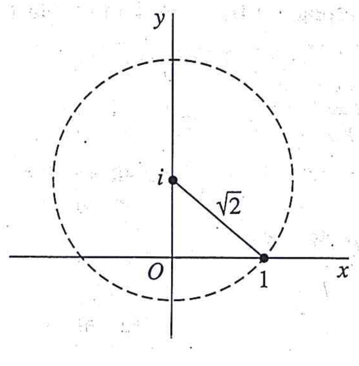

FIGURE 79
$|z - i| < \sqrt{2}$

and we arrive at the Taylor series expansion

$$\frac{1}{1-z} = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

EXAMPLE 2. Since the function $f(z) = e^z$ is entire, it has a Maclaurin series representation that is valid for all z. Here $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, \ldots)$; and because $f^{(n)}(0) = 1 (n = 0, 1, 2, \ldots)$, expansion (2) follows. Note that if $z = x + i0$, the expansion becomes

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

The entire function $z^3 e^{2z}$ is also represented by a Maclaurin series. The simplest way to show this is to replace z by $2z$ in expression (2) and then multiply through the result by z^3:

$$z^3 e^{2z} = \sum_{n=0}^{\infty}\frac{2^n}{n!}z^{n+3} \qquad (|z| < \infty).$$

Finally, if we replace n by $n-3$ here, we have

$$z^3 e^{2z} = \sum_{n=3}^{\infty}\frac{2^{n-3}}{(n-3)!}z^n \qquad (|z| < \infty).$$

EXAMPLE 3. One can use expansion (2) and the definition (Sec. 37)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to find the Maclaurin series for the entire function $f(z) = \sin z$. To give the details, we refer to expansion (1) and write

$$\sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}[1-(-1)^n]\frac{i^n z^n}{n!} \qquad (|z| < \infty).$$

But $1 - (-1)^n = 0$ when n is even, and so we can replace n by $2n+1$ in this last series:

$$\sin z = \frac{1}{2i}\sum_{n=0}^{\infty}[1-(-1)^{2n+1}]\frac{i^{2n+1}z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

Inasmuch as

$$1 - (-1)^{2n+1} = 2 \quad \text{and} \quad i^{2n+1} = (i^2)^n i = (-1)^n i,$$

this reduces to expansion (3).

EXAMPLE 4. Using term by term differentiation, which will be justified in Sec. 71, we differentiate each side of equation (3) and write

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{d}{dz}z^{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{(2n+1)!}z^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$
$$(|z| < \infty).$$

Expansion (4) is now verified.

EXAMPLE 5. Because $\sinh z = -i\sin(iz)$, as pointed out in Sec. 39, we need only recall expansion (3) for $\sin z$ and write

$$\sinh z = -i\sum_{n=0}^{\infty}(-1)^n\frac{(iz)^{2n+1}}{(2n+1)!} \qquad (|z| < \infty),$$

which becomes

$$\sinh z = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

EXAMPLE 6. Since $\cosh z = \cos(iz)$, according to Sec. 39, the Maclaurin series (4) for $\cos z$ reveals that

$$\cosh z = \sum_{n=0}^{\infty}(-1)^n\frac{(iz)^{2n}}{(2n)!} \qquad (|z| < \infty),$$

and we arrive at the Maclaurin series representation

$$\cosh z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} \qquad (|z| < \infty).$$

Observe that the Taylor series for $\cosh z$ about the point $z_0 = -2\pi i$, for example, is obtained by replacing the variable z on each side of this last equation by $z + 2\pi i$ and then recalling (Sec. 39) that $\cosh(z + 2\pi i) = \cosh z$ for all z:

$$\cosh z = \sum_{n=0}^{\infty}\frac{(z+2\pi i)^{2n}}{(2n)!} \qquad (|z| < \infty).$$

65. NEGATIVE POWERS OF $(z - z_0)$

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem there. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$. Such series are extremely important and are taken up in the next section. They are often obtained by using one or more of the six Maclaurin series listed at the beginning of Sec. 64. In order that the reader be accustomed to series involving negative powers of $(z - z_0)$, we pause here with several examples before exploring their general theory.

68. EXAMPLES

The coefficients in a Laurent series are generally found by means other than appealing directly to the integral representations in Laurent's theorem (Sec. 66). This has already been illustrated in Sec. 65, where the series found were actually Laurent series. The reader is encouraged to go back to Sec. 65, as well as to Exercises 10 and 11 of that section, in order to see how in each case the punctured plane or disk in which the series is valid can now be predicted by Laurent's theorem. Also, we shall always assume that the Maclaurin series expansions (1) through (6) in Sec. 64 are well known, since we shall need them so often in finding Laurent series. As was the case with Taylor series, we defer the proof of uniqueness of Laurent series till Sec. 72.

EXAMPLE 1. The function

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} \cdot \frac{1}{1+z^2}$$

has singularities at the points $z = 0$ and $z = \pm i$. Let us find the Laurent series representation of $f(z)$ that is valid in the punctured disk $0 < |z| < 1$ (see Fig. 82).

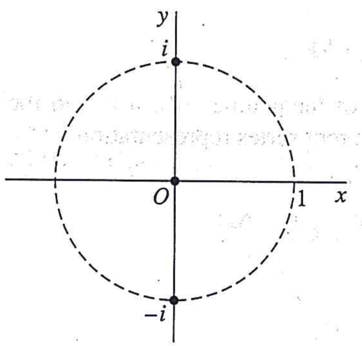

FIGURE 82

Since $|-z^2| < 1$ when $|z| < 1$, we may substitute $-z^2$ for z in the Maclaurin series expansion

(1) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

The result is

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z| < 1),$$

and so

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \qquad (0 < |z| < 1).$$

That is,

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1} \qquad (0 < |z| < 1).$$

Replacing n by $n+1$, we arrive at

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} \qquad (0 < |z| < 1).$$

In standard form, then,

(2) $$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z} \qquad (0 < |z| < 1).$$

(See also Exercise 3.)

EXAMPLE 2. The function

$$f(z) = \frac{z+1}{z-1},$$

which has the singular point $z = 1$, is analytic in the domains (Fig. 83)

$$D_1 : |z| < 1 \quad \text{and} \quad D_2 : 1 < |z| < \infty.$$

In these domains $f(z)$ has series representations in powers of z. Both series can be found by making appropriate replacements for z in the same expansion (1) that was used in Example 1.

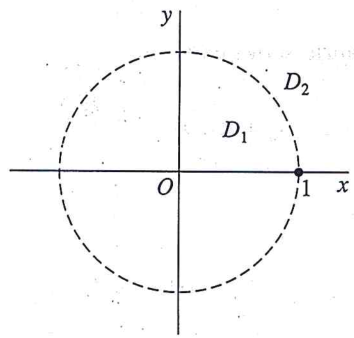

FIGURE 83

We consider first the domain D_1 and note that the series asked for is a Maclaurin series. In order to use series (1), we write

$$f(z) = -(z+1)\frac{1}{1-z} = -z\frac{1}{1-z} - \frac{1}{1-z}.$$

Then

$$f(z) = -z\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

Replacing n by $n-1$ in the first of the two series on the far right here yields the desired Maclaurin series:

$$f(z) = -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -1 - 2\sum_{n=1}^{\infty} z^n \qquad (|z| < 1). \tag{3}$$

The representation of $f(z)$ in the unbounded domain D_2 is a Laurent series, and the fact that $|1/z| < 1$ when z is a point in D_2 suggests that we use series (1) to write

$$f(z) = \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} = \left(1 + \frac{1}{z}\right)\frac{1}{1 - \frac{1}{z}} = \left(1 + \frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$
$$(1 < |z| < \infty).$$

Substituting $n - 1$ for n in the last of these series reveals that

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1 < |z| < \infty),$$

and we arrive at the Laurent series

$$f(z) = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1 < |z| < \infty). \tag{4}$$

EXAMPLE 3. Replacing z by $1/z$ in the Maclaurin series expansion

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we have the Laurent series representation

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \qquad (0 < |z| < \infty).$$

Note that no positive powers of z appear here, since the coefficients of the positive powers are zero. Note, too, that the coefficient of $1/z$ is unity; and, according to Laurent's theorem in Sec. 66, that coefficient is the number

$$b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\,dz$$

where C is any positively oriented simple closed contour around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed contours will be developed in considerable detail in Chap. 6 and then used extensively in Chap. 7.

EXAMPLE 4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n(z-i)^n \qquad (0 < |z - i| < \infty)$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From expression (5), Sec. 66, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm1, \pm2, \ldots)$$

where C is, for instance, any positively oriented circle $|z - i| = R$ about the point $z_0 = i$. Thus [compare with Exercise 13, Sec. 46]

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2, \\ 2\pi i & \text{when } n = -2. \end{cases}$$

EXERCISES

1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Ans. $1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}$.

2. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1 + (1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}$.

3. Find the Laurent series that represents the function $f(z)$ in Example 1, Sec. 68, when $1 < |z| < \infty$.

Ans. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$.

(*b*) Show that the linear fractional transformation

$$w = \frac{z-2}{z}$$

can be written

$$Z = z - 1, \quad W = i\frac{1-Z}{1+Z}, \quad w = iW.$$

Then, with the aid of the result in part (*a*), verify that it maps the disk $|z-1| \leq 1$ onto the left half plane $\text{Re}\, w \leq 0$.

4. Transformation (1), Sec. 102, maps the point $z = \infty$ onto the point $w = \exp(i\alpha)$, which lies on the boundary of the disk $|w| \leq 1$. Show that if $0 < \alpha < 2\pi$ and the points $z = 0$ and $z = 1$ are to be mapped onto the points $w = 1$ and $w = \exp(i\alpha/2)$, respectively, the transformation can be written

$$w = e^{i\alpha}\left[\frac{z + \exp(-i\alpha/2)}{z + \exp(i\alpha/2)}\right].$$

5. Note that when $\alpha = \pi/2$, the transformation in Exercise 4 becomes

$$w = \frac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}.$$

Verify that this special case maps points on the x axis as indicated in Fig. 123.

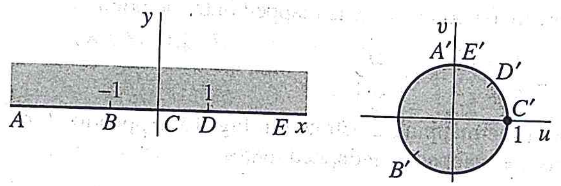

FIGURE 123
$w = \dfrac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}$.

6. Show that if $\text{Im}\, z_0 < 0$, transformation (1), Sec. 102, maps the lower half plane $\text{Im}\, z \leq 0$ onto the unit disk $|w| \leq 1$.

7. The equation $w = \log(z-1)$ can be written

$$Z = z - 1, \quad w = \log Z.$$

Find a branch of $\log Z$ such that the cut z plane consisting of all points except those on the segment $x \geq 1$ of the real axis is mapped by $w = \log(z-1)$ onto the strip $0 < v < 2\pi$ in the w plane.

103. MAPPINGS BY THE EXPONENTIAL FUNCTION

The object of this section is to provide the reader with some examples of mappings by the exponential function e^z that was introduced in Chap. 3 (Sec. 30). Our examples are reasonably simple, and we begin here by examining the images of vertical and horizontal lines.

EXAMPLE 1. We know from Sec. 30 that the transformation

$$(1) \qquad w = e^z$$

can be written $w = e^x e^{iy}$, where $z = x + iy$. Thus, if $w = \rho e^{i\phi}$,

$$(2) \qquad \rho = e^x, \quad \phi = y.$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w plane. That image moves counterclockwise around the circle shown in Fig. 124 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Consequently, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 124.

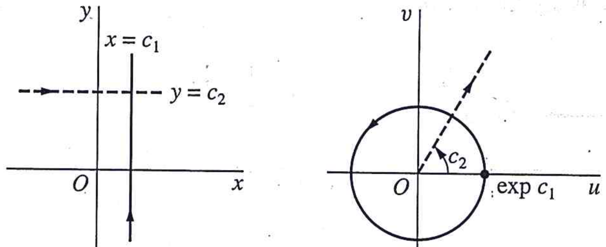

FIGURE 124
$w = \exp z$.

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

EXAMPLE 2. Let us show that the transformation $w = e^z$ maps the rectangular region $a \leq x \leq b, c \leq y \leq d$ onto the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$. The two regions and corresponding parts of their boundaries are indicated in Fig. 125.

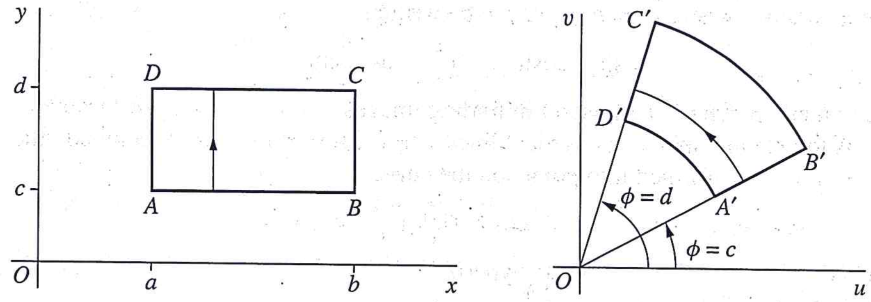

FIGURE 125
$w = \exp z$.

The vertical line segment AD is mapped onto the arc $\rho = e^a$, $c \le \phi \le d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b$, $c \le \phi \le d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \le \phi \le \pi$; and the rectangular region is mapped onto half of a circular ring, as shown in Fig. 8, Appendix 2.

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w plane (Fig. 126). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

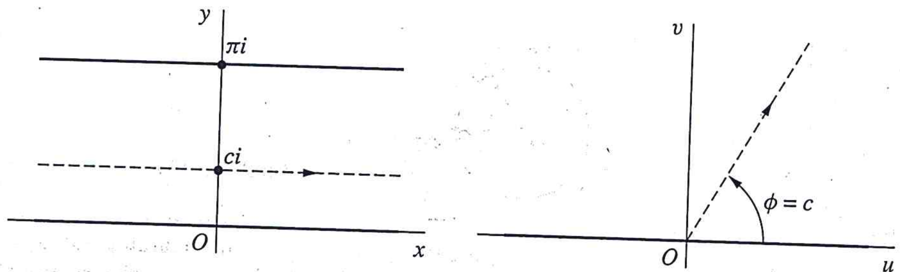

FIGURE 126
$w = \exp z$.

104. MAPPING VERTICAL LINE SEGMENTS BY $w = \sin z$

Since (Sec. 37) $\sin z = \sin x \cosh y + i \cos x \sinh y$, where $z = x + iy$, the transformation $w = \sin z$, where $w = u + iv$, can be written

(1) $$u = \sin x \cosh y, \quad v = \cos x \sinh y.$$

One method that is often useful in finding images of regions under this transformation is to examine images of vertical lines $x = c_1$. If $0 < c_1 < \pi/2$, points on the line $x = c_1$ are transformed into points on the curve

(2) $$u = \sin c_1 \cosh y, \quad v = \cos c_1 \sinh y \qquad (-\infty < y < \infty),$$

which is the right-hand branch of the hyperbola

(3) $$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1$$

with foci at the points

$$w = \pm\sqrt{\sin^2 c_1 + \cos^2 c_1} = \pm 1.$$

The second of equations (2) shows that as a point (c_1, y) moves upward along the entire length of the line, its image moves upward along the entire length of the hyperbola's branch. Such a line and its image are shown in Fig. 127, where corresponding points are labeled. Note that, in particular, there is a one to one mapping of the top half $(y > 0)$ of the line onto the top half $(v > 0)$ of the hyperbola's branch. If $-\pi/2 < c_1 < 0$, the line $x = c_1$ is mapped onto the left-hand branch of the same hyperbola. As before, corresponding points are indicated in Fig. 127.

The line $x = 0$, or the y axis, needs to be considered separately. According to equations (1), the image of each point $(0, y)$ is $(0, \sinh y)$. Hence the y axis is mapped onto the v axis in a one to one manner, the positive y axis corresponding to the positive v axis.

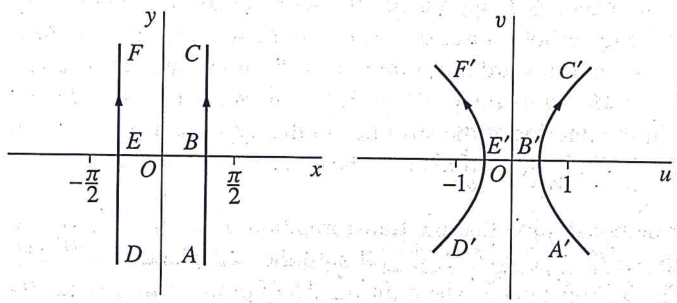

FIGURE 127
$w = \sin z$.

We now illustrate how these observations can be used to establish the images of certain regions.

EXAMPLE. Here we show that the transformation $w = \sin z$ is a one to one mapping of the semi-infinite strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ in the z plane onto the upper half $v \ge 0$ of the w plane.

To do this, we first show that the boundary of the strip is mapped in a one to one manner onto the real axis in the w plane, as indicated in Fig. 128. The image of the line segment BA there is found by writing $x = \pi/2$ in equations (1) and restricting y to be nonnegative. Since $u = \cosh y$ and $v = 0$ when $x = \pi/2$, a typical point $(\pi/2, y)$ on BA is mapped onto the point $(\cosh y, 0)$ in the w plane; and that image must move to the right from B' along the u axis as $(\pi/2, y)$ moves upward from B. Points $(x, 0)$ on the horizontal segment DB have images $(\sin x, 0)$, which move to the right from D' to B' as x increases from $x = -\pi/2$ to $x = \pi/2$, or as $(x, 0)$ goes from D to B. Finally, as points $(-\pi/2, y)$ on the line segment DE move upward from D, their images $(-\cosh y, 0)$ move left from D'.

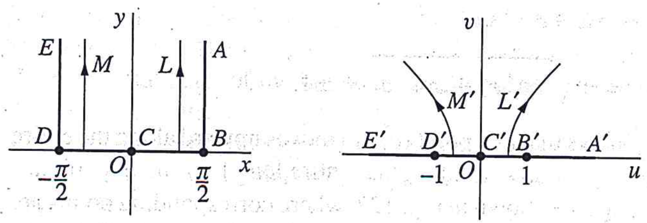

FIGURE 128
$w = \sin z$.

Now each point in the interior $-\pi/2 < x < \pi/2$, $y > 0$ of the strip lies on one of the vertical half lines $x = c_1$, $y > 0$ $(-\pi/2 < c_1 < \pi/2)$ that are shown in Fig. 128. Also, it is important to notice that the images of those half lines are distinct and constitute the entire half plane $v > 0$. More precisely, if the upper half L of a line $x = c_1$ $(0 < c_1 < \pi/2)$ is thought of as moving to the left toward the positive y axis, the right-hand branch of the hyperbola containing its image L' is opening up wider and its vertex $(\sin c_1, 0)$ is tending toward the origin $w = 0$. Hence L' tends to become the positive v axis, which we saw just prior to this example is the image of the positive y axis. On the other hand, as L approaches the segment BA of the boundary of the strip, the branch of the hyperbola closes down around the segment $B'A'$ of the u axis and its vertex $(\sin c_1, 0)$ tends toward the point $w = 1$. Similar statements can be made regarding the half line M and its image M' in Fig. 128. We may conclude that the image of each point in the interior of the strip lies in the upper half plane $v > 0$ and, furthermore, that each point in the half plane is the image of exactly one point in the interior of the strip.

This completes our demonstration that the transformation $w = \sin z$ is a one to one mapping of the strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ onto the half plane $v \ge 0$. The final result is shown in Fig. 9, Appendix 2. The right-hand half of the strip is evidently mapped onto the first quadrant of the w plane, as shown in Fig. 10, Appendix 2.

105. MAPPING HORIZONTAL LINE SEGMENTS BY $w = \sin z$

Another convenient way to find the images of certain regions when $w = \sin z$ is to consider the images of *horizontal* line segments $y = c_2$ $(-\pi \le x \le \pi)$, where $c_2 > 0$. According to equations (1) in Sec. 104, the image of such a line segment is the curve with parametric representation

(1) $$u = \sin x \cosh c_2, \quad v = \cos x \sinh c_2 \qquad (-\pi \le x \le \pi).$$

That curve is readily seen to be the ellipse

(2) $$\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1,$$

whose foci lie at the points

$$w = \pm\sqrt{\cosh^2 c_2 - \sinh^2 c_2} = \pm 1.$$

The image of a point (x, c_2) moving to the right from point A to point E in Fig. 129 makes one circuit around the ellipse in the clockwise direction. Note that when smaller values of the positive number c_2 are taken, the ellipse becomes smaller but retains the same foci $(\pm 1, 0)$. In the limiting case $c_2 = 0$, equations (1) become

$$u = \sin x, \quad v = 0 \qquad (-\pi \le x \le \pi);$$

and we find that the interval $-\pi \le x \le \pi$ of the x axis is mapped onto the interval $-1 \le u \le 1$ of the u axis. The mapping is not, however, one to one, as it is when $c_2 > 0$.

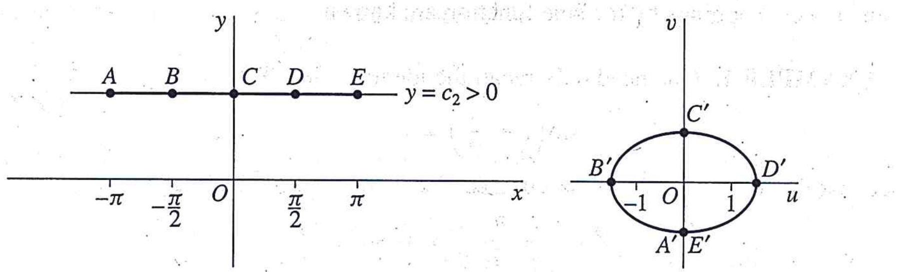

FIGURE 129
$w = \sin z$.

EXAMPLE. The rectangular region $-\pi/2 \le x \le \pi/2$, $0 \le y \le b$ is mapped by $w = \sin z$ in a one to one manner onto the semi-elliptical region that is shown in Fig. 130, where corresponding boundary points are also indicated. For if L is a line segment $y = c_2$ $(-\pi/2 \le x \le \pi/2)$, where $0 < c_2 \le b$, its image L' is the top half of the ellipse (2). As c_2 decreases, L moves downward toward the x axis and the semi-ellipse L' also moves downward and tends to become the line segment $E'F'A'$ from $w = -1$ to $w = 1$. In fact, when $c_2 = 0$, equations (1) become

$$u = \sin x, \quad v = 0 \qquad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right);$$

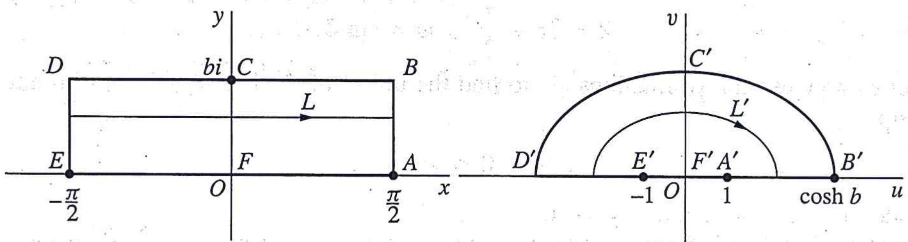

FIGURE 130
$w = \sin z$.

and this is clearly a one to one mapping of the segment EFA onto $E'F'A'$. Inasmuch as any point in the semi-elliptical region in the w plane lies on one and only one of the semi-ellipses, or on the limiting case $E'F'A'$, that point is the image of exactly one point in the rectangular region in the z plane. The desired mapping, which is also shown in Fig. 11 of Appendix 2, is now established.

106. SOME RELATED MAPPINGS

Mappings by various other functions closely related to the sine function are easily obtained once mappings by the sine function are known.

EXAMPLE 1. One need only recall the identity (Sec. 37)

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z$$

to see that the transformation $w = \cos z$ can be written successively as

$$Z = z + \frac{\pi}{2}, \quad w = \sin Z.$$

Hence the cosine transformation is the same as the sine transformation preceded by a translation to the right through $\pi/2$ units.

EXAMPLE 2. According to Sec. 39, the transformation $w = \sinh z$ can be written $w = -i\sin(iz)$, or

$$Z = iz, \quad W = \sin Z, \quad w = -iW.$$

It is, therefore, a combination of the sine transformation and rotations through right angles. The transformation $w = \cosh z$ is, likewise, essentially a cosine transformation since $\cosh z = \cos(iz)$.

EXAMPLE 3. With the aid of the identities

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z \quad \text{and} \quad \cos(iz) = \cosh z$$

that were used in the two examples just above, one can write the transformation $w = \cosh z$ as

$$Z = iz + \frac{\pi}{2}, \quad w = \sin Z. \tag{1}$$

Let us now use transformations (1) to find the image of the horizontal semi-infinite strip

$$x \geq 0, \; 0 \leq y \leq \pi/2$$

under the transformation $w = \cosh z$.

The first of transformations (1) is a rotation of the given strip through a right angle in the positive direction followed by a translation $\pi/2$ units to the right, as shown in Fig. 131. The transformation $w = \sin Z$ then maps the resulting strip onto the first

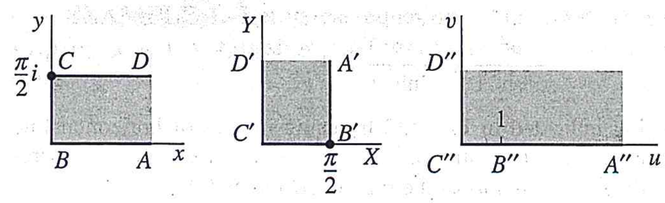

FIGURE 131
$w = \cosh z$.

quadrant of the w plane, as pointed out at the end of Sec. 104 and shown in Fig. 10, Appendix 2. It is left to the reader to verify corresponding boundary points of the given strip and the first quadrant that are labeled in Fig. 131.

EXERCISES

1. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho\exp(i\phi)$.
2. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \leq x \leq b, c \leq y \leq d$ under the transformation $w = \exp z$ is the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$, as shown in Fig. 125 (Sec. 103).
3. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.
4. Find the image of the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.
5. Show that the transformation $w = \sin z$ maps the top half $(y > 0)$ of the vertical line $x = c_1$ $(-\pi/2 < c_1 < 0)$ in a one to one manner onto the top half $(v > 0)$ of the left-hand branch of hyperbola (3), Sec. 104, as indicated in Fig. 128 of that section.
6. Show that under the transformation $w = \sin z$, a line $x = c_1$ $(\pi/2 < c_1 < \pi)$ is mapped onto the right-hand branch of hyperbola (3), Sec. 104. Note that the mapping is one to one and that the upper and lower halves of the line are mapped onto the *lower* and *upper* halves, respectively, of the branch.
7. Vertical half lines were used in the example in Sec. 104 to show that the transformation $w = \sin z$ is a one to one mapping of the open region $-\pi/2 < x < \pi/2, y > 0$ onto the half plane $v > 0$. Verify that result by using, instead, the *horizontal* line segments $y = c_2$ $(-\pi/2 < x < \pi/2)$, where $c_2 > 0$.
8. (*a*) Show that under the transformation $w = \sin z$, the images of the line segments forming the boundary of the rectangular region $0 \leq x \leq \pi/2, 0 \leq y \leq 1$ are the line segments and the arc $D'E'$ shown in Fig. 132. The arc $D'E'$ is a quarter of the

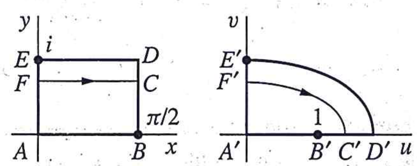

FIGURE 132
$w = \sin z$.

ellipse

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1.$$

(*b*) Complete the mapping indicated in Fig. 132 by using images of horizontal line segments to prove that the transformation $w = \sin z$ establishes a one to one correspondence between the interior points of the regions $ABDE$ and $A'B'D'E'$.

9. Verify that the interior of a rectangular region $-\pi \le x \le \pi, a \le y \le b$ lying above the x axis is mapped by $w = \sin z$ onto the interior of an elliptical ring which has a cut along the segment $-\sinh b \le v \le -\sinh a$ of the negative imaginary axis, as indicated in Fig. 133. Note that while the mapping of the interior of the rectangular region is one to one, the mapping of its boundary is *not*.

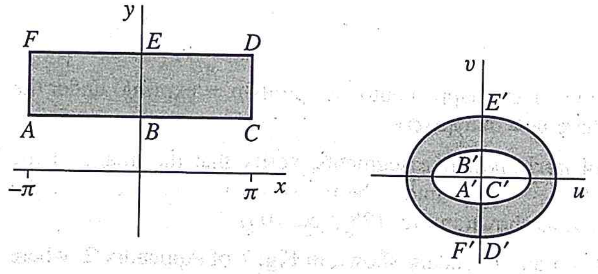

FIGURE 133
$w = \sin z$.

10. Observe that the transformation $w = \cosh z$ can be expressed as a composition of the mappings

$$Z = e^z, \quad W = Z + \frac{1}{Z}, \quad w = \frac{1}{2}W.$$

Then, by referring to Figs. 7 and 16 in Appendix 2, show that when $w = \cosh z$, the semi-infinite strip $x \le 0, 0 \le y \le \pi$ in the z plane is mapped onto the lower half $v \le 0$ of the w plane. Indicate corresponding parts of the boundaries.

11. (*a*) Verify that the equation $w = \sin z$ can be written

$$Z = i\left(z + \frac{\pi}{2}\right), \quad W = \cosh Z, \quad w = -W.$$

(*b*) Use the result in part (*a*) here and the one in Exercise 10 to show that the transformation $w = \sin z$ maps the semi-infinite strip $-\pi/2 \le x \le \pi/2, y \ge 0$ onto the half plane $v \ge 0$, as shown in Fig. 9, Appendix 2. (This mapping was verified in a different way in the example in Sec. 104 and in Exercise 7.)

107. MAPPINGS BY z^2

In Chap 2 (Sec. 14), we considered some fairly simple mappings under the transformation $w = z^2$, written in the form

$$u = x^2 - y^2, \quad v = 2xy. \tag{1}$$

We turn now to a less elementary example and then (Sec. 108) examine related mappings $w = z^{1/2}$, where specific branches of the square root function are taken.

EXAMPLE 1. Let us use equations (1) to show that the image of the vertical strip $0 \le x \le 1$, $y \ge 0$, shown in Fig. 134, is the closed semiparabolic region indicated there.

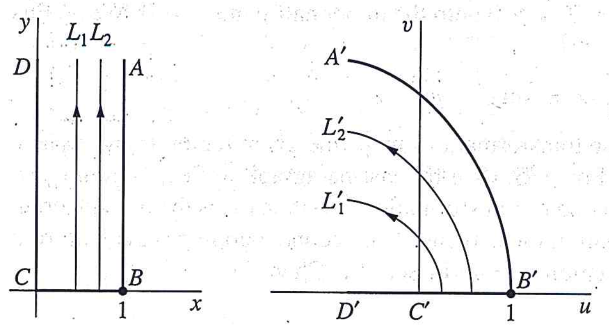

FIGURE 134
$w = z^2$.

When $0 < x_1 < 1$, the point (x_1, y) moves up a vertical half line, labeled L_1 in Fig. 134, as y increases from $y = 0$. The image traced out in the uv plane has, according to equations (1), the parametric representation

$$u = x_1^2 - y^2, \quad v = 2x_1 y \quad (0 \le y < \infty). \tag{2}$$

Using the second of these equations to substitute for y in the first one, we see that the image points (u, v) must lie on the parabola

$$v^2 = -4x_1^2(u - x_1^2), \tag{3}$$

with vertex at $(x_1^2, 0)$ and focus at the origin. Since v increases with y from $v = 0$, according to the second of equations (2), we also see that as the point (x_1, y) moves up L_1 from the x axis, its image moves up the top half L_1' of the parabola from the u axis. Furthermore, when a number x_2 larger than x_1 but less than 1 is taken, the corresponding half line L_2 has an image L_2' that is a half parabola to the right of L_1', as indicated in Fig. 134. We note, in fact, that the image of the half line BA in that figure is the top half of the parabola $v^2 = -4(u - 1)$, labeled $B'A'$.

The image of the half line CD is found by observing from equations (1) that a typical point $(0, y)$, where $y \ge 0$, on CD is transformed into the point $(-y^2, 0)$ in the uv plane. So, as a point moves up from the origin along CD, its image moves left from the origin along the u axis. Evidently, then, as the vertical half lines in the xy plane move to the left, the half parabolas that are their images in the uv plane shrink down to become the half line $C'D'$.

It is now clear that the images of all the half lines between and including CD and BA fill up the closed semiparabolic region bounded by $A'B'C'D'$. Also, each point in that region is the image of only one point in the closed strip bounded by $ABCD$. Hence we may conclude that the semiparabolic region is the image of the strip and that there is a one to one correspondence between points in those closed regions. (Compare with Fig. 3 in Appendix 2, where the strip has arbitrary width.)